

INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
POLISH ACADEMY OF SCIENCES

Andrzej Ziolkowski

**On Symmetric Tensors
of Second and Fourth Order
in Mechanics**

with focus on Cauchy stress and Hooke tensors

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